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Mean-Field Bounds on the Magnetization for Ferromagnetic Spin Models

Paul A. Pearce[1]

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The mean-field approximation is shown to give an upper bound on the magnetization for a large class of one-component models with arbitrary ferromagnetic pair interactions. Specific examples include discrete and continuous spin Ising models. In addition, a new comparison inequality for multicomponent rotators is proven which allows this result to be extended to the plane rotator and classical Heisenberg ferromagnets.

KEY WORDS: Mean-field theory; correlation inequalities; Ising models; vector spin models.

1. INTRODUCTION

Mean-field models are generally regarded as a somewhat crude first approximation to ferromagnetic spin models. Even so, the value of the mean-field approximation is greatly enhanced in those circumstances in which it places strict bounds on the quantities of interest, such as, the critical temperature and magnetization. The first bound of this kind was obtained by Griffiths.[1] Using a special correlation inequality for the spin-1/2 Ising ferromagnet, he showed that the onset of spontaneous magnetization always occurs at a temperature below the mean-field critical value, that is, $T_c \leqslant T_{\mathrm{MF}}$. The same result has also been obtained recently[2] for a large class of one-component (Ising) models by the use of Dobrushin's uniqueness theorem, and more recently,[3,4] for the n-vector model by the use of local Ward identities.

Much less is known about mean-field bounds on the magnetization. The only result so far, for the spin-1/2 Ising ferromagnet, is due to

[1]The Institute for Advanced Study, Princeton, New Jersey 08540. Present address: Research School of Physical Sciences, The Australian National University, Canberra, Australia.

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Thompson.[5] Using specific properties of this model, he showed that the magnetization is always bounded above by the mean-field magnetization. This remarkable result immediately implies the mean-field bound on the critical temperature. A natural question to ask then is: Does this stronger bound hold for models other than the spin-1/2 Ising model?

In this paper new methods are developed to obtain mean-field bounds on the magnetization. In particular, the new methods apply to generalized Ising and vector spin models. The remaining two parts of this introduction are devoted to defining the models and discussing their mean-field counterparts. In Section 2 the results for one-component (Ising) models are presented. A new comparison inequality for multicomponent rotators is proved in Section 3. This is then combined with the results for one-component models to obtain mean-field bounds on the magnetization for the plane rotator and classical Heisenberg models. Finally, in Section 4, a general theorem is proved giving mean-field bounds on critical temperatures.

It is perhaps worth mentioning at this point that mean-field upper bounds on the magnetization are the best possible for arbitrary ferromagnetic pair interactions. It is known, for example, that equality can be attained in the limit of long-range interactions. Nevertheless, by restricting Thompson's methods to the case of *nearest-neighbor* interactions on a regular lattice with a *fixed* coordination number, Krinsky[6] has managed to obtain an improved bound on the magnetization for the spin-1/2 Ising model, corresponding to the Bethe approximation.

1.1. The Spin Models

Let Λ be a finite periodic lattice. To each site $i \in \Lambda$ associate either a one-component (Ising) spin $\sigma_i \in \mathbb{R}$ or an n-dimensional unit vector (classical spin) $\mathbf{S}_i = (S_i^1, S_i^2, \ldots, S_i^n) \in S^n$, where S^n denotes the unit sphere in $\mathbb{R}^n$. The model Hamiltonians to be considered are then

$$H(\sigma) = -\frac{1}{2} \sum_{i,j\in\Lambda} J_{ij}\sigma_i\sigma_j - h \sum_{i\in\Lambda} \sigma_i \tag{1}$$

and

$$H(\mathbf{S}) = -\frac{1}{2} \sum_{i,j\in\Lambda} J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j - h \sum_{i\in\Lambda} S_i^1 \tag{2}$$

Unless it is explicitly stated to the contrary in the sequel, it will tacitly be assumed that the interactions for these models are ferromagnetic and translationally invariant, so that

$$J_{ij} = J(i-j) \geqslant 0, \qquad h \geqslant 0 \tag{3}$$

and

$$J = \sum_{j \in \Lambda} J_{ij} \tag{4}$$

independent of i in Λ.

To complete the definition of the models, it is necessary to specify *a priori* measures on the single-spin spaces $\mathbb{R}$ and S^n. For the one-component model the probability measure on $\mathbb{R}$ will be denoted ν and called the single-spin measure. For the n-vector model, the probability measure will always be the (normalized) uniform measure on the sphere S^n. The configuration spaces are, respectively,

$$\{\sigma\} = \bigotimes_{i \in \Lambda} \mathbb{R}, \qquad \{\mathbf{S}\} = \bigotimes_{i \in \Lambda} S^n \tag{5}$$

For the one-component model, with single-spin measure ν, the expected value of an observable $f(\sigma)$ will then be

$$\langle f(\sigma) \rangle = \int_{\{\sigma\}} f(\sigma) e^{-\beta H(\sigma)} \prod_{i \in \Lambda} d\nu(\sigma_i) \bigg/ \int_{\{\sigma\}} e^{-\beta H(\sigma)} \prod_{i \in \Lambda} d\nu(\sigma_i) \tag{6}$$

where $\beta = 1/k_B T$ is the inverse temperature and the integrals extend over the entire configuration space. Similarly, for the n-vector model (2),

$$\langle f(\mathbf{S}) \rangle = \int_{\{\mathbf{S}\}} f(\mathbf{S}) e^{-\beta H(\mathbf{S})} \prod_{i \in \Lambda} d\mathbf{S}_i \bigg/ \int_{\{\mathbf{S}\}} e^{-\beta H(\mathbf{S})} \prod_{i \in \Lambda} d\mathbf{S}_i \tag{7}$$

For convenience in the sequel, the temperature dependence will usually be suppressed (i.e., the factor β will be absorbed into the interactions J_{ij} and h).

1.2. Mean-Field Theory

The mean-field or equivalent-neighbor models are defined by setting all the interactions $J_{ij} = J/|\Lambda|$, where $|\Lambda|$ denotes the number of sites in Λ. With this replacement, the models (1) and (2) can be solved exactly. For definiteness, consider the spin-s Ising model given by the Hamiltonian (1) with the single-spin measure

$$d\nu(\sigma) = \frac{1}{2s+1} \sum_{l=-s}^{s} \delta(\sigma + l/s)\, d\sigma \tag{8}$$

In this case the solution of the mean-field model is well known[7,8] and, in particular, the mean-field magnetization m is given as the largest root of the equation

$$m = B_s(Jm + h) \tag{9}$$

where

$$B_s(x) = \frac{2s+1}{2s}\coth\left(\frac{2s+1}{2s}x\right) - \frac{1}{2s}\coth\left(\frac{1}{2s}x\right) \tag{10}$$

is the Brillouin function.

More generally, the mean-field magnetization $m = m(J,h)$ will be the largest root of the equation

$$m = \int d\nu(\sigma)\sigma e^{k\sigma} \Big/ \int d\nu(\sigma) e^{k\sigma} \tag{11}$$

with

$$k = Jm + h \tag{12}$$

This certainly holds, for example, if $\nu \in \mathfrak{M}$; where $\mathfrak{M}$ is the class of even probability measures on $\mathbb{R}$ with compact support such that $m(k)$, as given by (11), is concave for $k \geqslant 0$. Under these circumstances, the function $m(k)$ looks qualitatively like the hyperbolic tangent. It is a real-analytic, bounded, and odd function of k; moreover, for $k \geqslant 0$ it is concave, nonnegative and nondecreasing with slope λ at the origin given by

$$\lambda = \int d\nu(\sigma)\sigma^2 \tag{13}$$

If $\nu \in \mathfrak{M}$ the mean-field theory exhibits spontaneous magnetization. For suppose $h > 0$. Then, from the properties listed above, it follows that (11) and (12) have a unique positive solution $m(J,h)$. Letting $h \to 0+$ it is found that

$$\lim_{h\to 0+} m(J,h) = \begin{cases} m(J,0) > 0, & J\lambda > 1 \\ 0, & J\lambda \leqslant 1 \end{cases} \tag{14}$$

Resurrecting the temperature dependence, it is seen that the critical temperature of the mean-field model is

$$T_{\mathrm{MF}} = k_B^{-1} J \int d\nu(\sigma)\sigma^2 \tag{15}$$

For the n-vector model (2), the mean-field magnetization is a vector pointing in the direction of the applied field (i.e., along the axis labeled 1) with magnitude m given[9] as the largest root of the equation

$$m = I_{(1/2)n}(Jm + h)/I_{(1/2)n-1}(Jm + h) \tag{16}$$

where $I_\mu(x)$ is the modified Bessel function of order μ. It is easily verified that this is just the mean-field equation [(11) with $k = Jm + h$] for a one-component model with single-spin measure on $[-1, 1]$ given by

$$d\nu(\sigma) = (1-\sigma^2)^{(1/2)(n-3)} d\sigma \Big/ \int_{-1}^{1} (1-\sigma^2)^{(1/2)(n-3)} d\sigma \tag{17}$$

As we shall see in Section 3, this is no coincidence!

2. ONE-COMPONENT BOUNDED SPINS

We show that the mean-field magnetization is an upper bound whenever the single-spin measure ν lies in a certain class $\mathcal{P}$. To define $\mathcal{P}$, let $m = m(k)$ depend on ν as prescribed by (11). The class $\mathcal{P}$ is then the set of all even probability measures on $\mathbb{R}$ with compact support, such that

$$\int d\nu(\sigma) e^{k\sigma}(m-\sigma)^p \geqslant 0 \tag{18}$$

whenever $p \in \mathbb{Z}^+$ and $k \geqslant 0$. Since the concavity of $m(k)$ for $k \geqslant 0$ is equivalent to condition (18) with $p = 3$, we see immediately that $\mathcal{P} \subset \mathfrak{M}$.

Theorem 1. Consider the one-component model with Hamiltonian (1) and suppose the single-spin measure $\nu \in \mathcal{P}$. Then

$$\langle \sigma_i \rangle \leqslant m \tag{19}$$

where $m = m(J, h) \geqslant 0$ is the largest root of (11) and (12).

Proof. Write

$$-H(\sigma) = \frac{1}{2} \sum_{i,j \in \Lambda} J_{ij}(m-\sigma_i)(m-\sigma_j) + (Jm+h)\sum_{i \in \Lambda} \sigma_i - \frac{1}{2}|\Lambda| J m^2 \tag{20}$$

and set

$$-H_0(\sigma) = (Jm+h)\sum_{i \in \Lambda} \sigma_i \tag{21}$$

Then

$$\langle m-\sigma_i \rangle = Z^{-1}\left\langle (m-\sigma_i)\exp\left[\frac{1}{2}\sum_{i,j \in \Lambda} J_{ij}(m-\sigma_i)(m-\sigma_j)\right]\right\rangle_0 \tag{22}$$

where

$$Z = \left\langle \exp\left[\frac{1}{2}\sum_{i,j \in \Lambda} J_{ij}(m-\sigma_i)(m-\sigma_j)\right]\right\rangle_0 \geqslant 0 \tag{23}$$

and $\langle \cdots \rangle_0$ denotes expectation (6) with respect to the Hamiltonian H_0. Now expand the exponential on the right side of (22). The result is a series of nonnegative terms because $\langle \cdots \rangle_0$ factors over the sites and, by hypothesis,

$$\left\langle (m-\sigma_i)^p \right\rangle_0 \geqslant 0 \tag{24}$$

for any $p \in \mathbb{Z}^+$ and $i \in \Lambda$. This proves (19). ■

This is the main theorem. The problem now is to determine what measures lie in $\mathcal{P}$. Unfortunately, we do not at present have a complete solution to this problem. It is possible, however, to show that many discrete and continuous-spin measures do lie in $\mathcal{P}$. We begin by proving the following general result.

Lemma 2. If $\nu_1(\sigma), \nu_2(\sigma) \in \mathcal{P}$, and $\alpha > 0$, then $\nu_1(\alpha\sigma)$ and $\nu_1 * \nu_2(\sigma) \in \mathcal{P}$. That is, $\mathcal{P}$ is closed under scale transformations, and convolutions[10] defined formally by

$$d(\nu_1 * \nu_2)(\sigma) = \int d\nu_1(\tau)\, d\nu_2(\sigma - \tau) \tag{25}$$

Proof. The first part is easily obtained by a change of variables. To prove closure under convolutions we define

$$\Phi(x) = \mathfrak{L}(\nu)(x) = \int d\nu(\sigma) e^{x\sigma} \tag{26}$$

to be the bilateral Laplace transform of ν and set

$$D(x) = \log \Phi(x) \tag{27}$$

Since Φ is entire, the condition (18) is seen to be equivalent to requiring that, for all $k \geqslant 0$, the function (primes denote differentiation)

$$\Phi(k - r)\exp\left[r\Phi'(k)/\Phi(k) \right] = \exp\left[D(k - r) + rD'(k) \right] \tag{28}$$

be strongly positive (i.e., have a Taylor expansion about $r = 0$ with coefficients that are all nonnegative). The result thus follows from the convolution theorem[10]

$$\mathfrak{L}(\nu_1 * \nu_2) = \mathfrak{L}(\nu_1)\mathfrak{L}(\nu_2) \tag{29}$$

and the fact that the product of two strongly positive functions is strongly positive. ■

Corollary 3. Consider the spin-s Ising model given by (1) and (8) and suppose that

$$2s + 1 = 2^q \cdot 3^r \tag{30}$$

for some choice of nonnegative integers q and r. Then

$$\langle \sigma_i \rangle \leqslant m \tag{31}$$

where $m = m(J, h)$ is the largest root of (9).

Proof. By Theorem 1 we need only show that $\nu \in \mathcal{P}$ for the prescribed values of s. Consider first $s = 1/2$. In this case the condition (18) for odd p reduces, after some algebra, to the inequality

$$e^{kp} \cdot e^{-k} \geqslant e^{-kp} \cdot e^{k} \tag{32}$$

which is obviously true for all $p \geqslant 1$ and $k \geqslant 0$. Next consider $s = 1$. In this case the condition (18) for odd p is equivalent to

$$(2e^k + 1)^p e^{-k} + (e^k - e^{-k})^p - (1 + 2e^{-k})^p e^k \geqslant 0 \tag{33}$$

For $p = 1$ equality holds. For $p \geqslant 2$ the left side is not less than

$$(2 + e^{-k})^p e^k - (1 + 2e^{-k})^p e^k \tag{34}$$

which is nonnegative for $k \geqslant 0$ as required. To show $\nu \in \mathcal{P}$ for the other values of s we convolute spin-1/2 and spin-1 measures and use Lemma 2. For example, if $s = 5/2$, we find

$$\begin{aligned} \nu = {} & (1/6)\left[\delta(\sigma + (3/5)) + \delta(\sigma - (3/5))\right] \\ & * \left[\delta(\sigma + (2/5)) + \delta(\sigma) + \delta(\sigma - (2/5))\right] d\sigma \in \mathcal{P} \quad \blacksquare \end{aligned} \tag{35}$$

Lemma 4. Let ν be an even probability measure with support on $[-1, 1]$, and suppose ν is absolutely continuous, i.e., $d\nu(\sigma) = f(\sigma)\,d\sigma$, with f nondecreasing on $[0, 1]$. Then $\nu \in \mathcal{P}$.

Proof. Given $0 \leqslant m \leqslant 1$ and $k \geqslant 0$, we show

$$\int_{-1}^{1} d\nu(\sigma) e^{k\sigma}\left[(m - \sigma)^p - (1 - m)^{p-1}(m - \sigma)\right] \geqslant 0 \tag{36}$$

for all *odd* integers p. But now, for $\sigma \leqslant 2m - 1$ the integrand is nonnegative. Moreover, after the substitution $x = \sigma - m$, the remaining integral over the range $2m - 1 \leqslant \sigma \leqslant 1$ takes the form

$$\begin{aligned} & \int_{m-1}^{1-m} d\nu(m + x) e^{k(m+x)}\left[(1 - m)^{p-1}x - x^p\right] \\ & \quad = e^{km}\int_0^{1-m} dx\left[e^{kx}f(m + x) - e^{-kx}f(m - x)\right] \\ & \qquad \times\left[(1 - m)^{p-1}x - x^p\right] \end{aligned} \tag{37}$$

This is also nonnegative under the assumed conditions on f. So this proves (36). If we now choose $m = m(k)$ given by (11), we obtain (18) for p odd; for p even (18) is trivial. $\blacksquare$

Corollary 5. Consider the spin-∞ Ising model given by (1) with single-spin measure $d\nu(\sigma) = (1/2)\,d\sigma$, the uniform measure on $[-1, 1]$. Then $\nu \in \mathcal{P}$ and

$$\langle \sigma_i \rangle \leqslant m \tag{38}$$

where $m = m(J, h)$ is the largest root of the equation

$$m = L(Jm + h) \tag{39}$$

with $L(x) = \coth x - x^{-1}$ the Langevin function.

Proof. The result follows immediately from Lemma 4 and Theorem 1, or from Corollary 3 by taking the limit $s \to \infty$. $\blacksquare$

In concluding this section, it should be pointed out that the above results are not entirely satisfactory. First, it has not been shown that the spin-s measure (8) lies in $\mathcal{P}$ for all values of the spin, though this seems to be a reasonable conjecture. Second, it is vital in discussing the n-vector model in the next section to know whether the single-spin measure (17) lies in $\mathcal{P}$. Lemma 4 applies when $n = 2$ or 3, but the question has not yet been settled for $n \geqslant 4$.

3. MULTICOMPONENT ROTATORS

The key to obtaining mean-field bounds on the magnetization for rotators is to consider the Ising-like n-vector model. The spins of this model are n-dimensional unit vectors but the interaction, given by

$$H_1(\mathbf{S}) = -\sum_{i,j\in\Lambda} J_{ij} S_i^1 S_j^1 - h \sum_{i\in\Lambda} S_i^1 \tag{40}$$

involves only the components along the field. Therefore, if we set $S_i^1 = \sigma_i$, the other components can be integrated out. In this way the model reduces to a one-component model with single-spin measure on $[-1, 1]$ given by

$$d\nu(\sigma) = \frac{A_{n-1}}{A_n}(1-\sigma^2)^{(1/2)(n-3)}\,d\sigma \tag{41}$$

where

$$A_n = 2\pi^{(1/2)n}/\Gamma[(1/2)n] \tag{42}$$

is the surface area of a unit n-dimensional sphere.

To make this identification precise, let A be a multiplicity function assigning a nonnegative integer $A(i)$ to each site i in Λ, and define

$$\sigma_A = \prod_{i\in\Lambda} \sigma_i^{A(i)}, \text{ etc.} \tag{43}$$

Then, for any multiplicity function A, the assertion is that

$$\langle S_A^1\rangle_1 \equiv \langle \sigma_A \rangle \tag{44}$$

where on the left the expectation [cf. (7)] is with respect to the vector Hamiltonian (40) and on the right the expectation [cf. (6)] is with respect to the one-component Hamiltonian (1) with single-spin measure given by (41).

In mean-field approximation ($J_{ij} = J/|\Lambda|$) the Ising-like n-vector model (40) has exactly the same solution (16) as the isotropic n-vector model (2). The crucial result then is the following comparison inequality.

Theorem 6. Consider the n-vector model (2) and suppose the interactions are ferromagnetic (i.e., $J_{ij} \geqslant 0$, $h \geqslant 0$; the J_{ij} need not be translation invariant). Then for any multiplicity function A

$$\langle S_A^1 \rangle \leqslant \langle S_A^1 \rangle_1 \tag{45}$$

where the expectation on the right is with respect to the Ising-like n-vector model (40).

Proof. We define[11] a mixed duplicate Hamiltonian

$$H(\mathbf{S}, \overline{\mathbf{S}}) = H(\mathbf{S}) + H_1(\overline{\mathbf{S}}) \tag{46}$$

Then, taking expectations in the doubled system,[11] the result we want to prove is

$$\langle \overline{S}_A^1 - S_A^1 \rangle \geqslant 0 \tag{47}$$

To this end we parametrize the spins as

$$\mathbf{S}_i = (\cos\theta_i, \sin\theta_i \mathbf{V}_i) \tag{48}$$

and

$$\overline{\mathbf{S}}_i = \left(\cos\bar{\theta}_i, \sin\bar{\theta}_i \overline{\mathbf{V}}_i\right) \tag{49}$$

where $0 \leqslant \theta_i$, $\bar{\theta}_i < \pi$, and $\mathbf{V}_i$ and $\overline{\mathbf{V}}_i$ are $(n-1)$-dimensional unit vectors (spins). Under this transformation the configurational integrals become

$$\int_{S^n} d\mathbf{S}_i \int_{S^n} d\overline{\mathbf{S}}_i = \int_0^\pi d\theta_i \int_0^\pi d\bar{\theta}_i \left(\sin\theta_i \sin\bar{\theta}_i\right)^{n-2} \int_{S^{n-1}} d\mathbf{V}_i \int_{S^{n-1}} d\overline{\mathbf{V}}_i \tag{50}$$

To prove (47) we note that the denominator is nonnegative and expand the Boltzmann factor $\exp[-H(\mathbf{S}, \overline{\mathbf{S}})]$ in the numerator. Then, by expanding the terms such as

$$\overline{S}_i^1 \overline{S}_j^1 + \mathbf{S}_i \cdot \mathbf{S}_j = \cos\bar{\theta}_i \cos\bar{\theta}_j + \cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j \mathbf{V}_i \cdot \mathbf{V}_j \tag{51}$$

and $\overline{S}_A^1 - S_A^1$, in the usual way,[11] the numerator can be written as a polynomial with nonnegative coefficients in the variables:

$$\left(\cos\bar{\theta}_i \pm \cos\theta_i\right), \left(\sin\theta_i \pm \sin\bar{\theta}_i\right), \mathbf{V}_i \cdot \mathbf{V}_j \tag{52}$$

Performing the integrals (50) term-by-term now leads to the desired result because such integrals are known to be nonnegative (see Theorem 2.1 and Lemma 3.1 of Ref. 11).

Corollary 7. Consider the n-vector model (2) with $n = 2$ or 3. Then

$$\langle S_i^1 \rangle \leqslant m \tag{53}$$

where $m = m(J, h)$ is the largest root of Eq. (16) which, for $n = 3$, takes the familiar form

$$m = L(Jm + h) \tag{54}$$

with $L(x) = \coth x - x^{-1}$ the Langevin function.

Proof. By Theorem 6 and the identification (44)

$$\langle S_i^1 \rangle \leqslant \langle \sigma_i \rangle \tag{55}$$

where the single-spin measure on the right is given by (41). But now, for $n = 2$ or 3, this measure is in $\mathcal{P}$ by Lemma 4. The result is thus obtained by using Theorem 1 and the integral formulas:

$$\int_{-1}^{1} (1 - \sigma^2)^{\mu - 1/2} e^{k\sigma} \, d\sigma = \frac{\pi^{1/2} \Gamma(\mu + (1/2)) I_\mu(k)}{((1/2)k)^\mu} \tag{56}$$

$$\int_{-1}^{1} (1 - \sigma^2)^{\mu - 1/2} \sigma e^{k\sigma} \, d\sigma = \frac{\pi^{1/2} \Gamma(\mu + (1/2)) I_{\mu+1}(k)}{((1/2)k)^\mu} \tag{57}$$

4. BOUNDS ON CRITICAL TEMPERATURES

For the one-component model (1), Cassandro *et al.*[2] have shown that $T_c \leqslant T_{\mathrm{MF}}$ whenever the single-spin measure $\nu \in \mathfrak{M}$. This is a very strong result, but it requires the full machinery of Dobrushin's uniqueness theorem and the Vasershtein distance. In this section we will prove a weaker, but it is hoped more accessible result. Namely, we prove that $T_c \leqslant T_{\mathrm{MF}}$ under the additional hypothesis that ν has the Lee–Yang property.[12] Since the single-spin measures of immediate interest here possess the Lee–Yang property, this is not a serious drawback.

Theorem 8. Consider the one-component model (1) with single-spin measure $\nu \in \mathfrak{M}$ and suppose that $\Phi(z)$, given by (26) for $z \in \mathbb{C}$, has only pure imaginary zeros. Then

$$T_c \leqslant T_{\mathrm{MF}} = k_B^{-1} J \int d\nu(\sigma) \sigma^2 \tag{58}$$

Proof. The Laplace transform $\Phi(z)$ is an entire function of order not exceeding 1. Therefore, by applying Hadamard's factorization theorem,[13] and using the assumed properties of ν and Φ we see that Φ can be written as an infinite product

$$\Phi(x) = \prod_{n=1}^{\infty} (1 + x^2/\alpha_n^2) \tag{59}$$

where the α_n can be ordered such that

$$0 < \alpha_1 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant \cdots . \tag{60}$$

Now consider an individual factor

$$\Phi_n(x) = 1 + x^2/\alpha_n^2 \tag{61}$$

For any such factor it is readily verified that the requisite function (28), viz.

$$\left[\alpha_n^2 + (k-r)^2\right]\exp\left[2kr/\left(\alpha_n^2 + k^2\right)\right] \tag{62}$$

is strongly positive for $0 \leqslant k \leqslant \alpha_n$. By forming products, it thus follows that the function (28), with $\Phi(x)$ given by (59), is strongly positive for $0 \leqslant k \leqslant \alpha_1$. Equivalently, we conclude that ν satisfies condition (18) for $0 \leqslant k \leqslant \alpha_1$. Now examining the proof of Theorem 1, we see that if $0 \leqslant Jm + h \leqslant \alpha_1$, where $m = m(J,h)$ is the largest root of (11) and (12), then

$$0 \leqslant \langle \sigma_i \rangle \leqslant m \tag{63}$$

Moreover, since α_1 does not depend on Λ, this inequality continues to hold in the thermodynamic limit. In particular, from (14) we see that

$$\lim_{h\to 0+} \lim_{\Lambda\to\infty} \langle \sigma_i \rangle = 0 \tag{64}$$

for $T \geqslant T_{\mathrm{MF}}$. That is, there is no spontaneous magnetization for $T \geqslant T_{\mathrm{MF}}$. Hence $T_c \leqslant T_{\mathrm{MF}}$. ■

Corollary 9. For the spin-s Ising model, given by (1) and (8),

$$T_c \leqslant k_B^{-1} J(s+1)/3s \tag{65}$$

Proof. The result follows from Theorem 8 because $\nu \in \mathfrak{M}$ and

$$\begin{aligned}\Phi(x) &= \frac{1}{2s+1}\,\frac{\sinh\left[(2s+1)x/2s\right]}{\sinh(x/2s)}\\ &= \prod_{n=1}^{\infty}\left\{1 + \left[\frac{(2s+1)x}{2n\pi s}\right]^2\right\} \Big/ \prod_{n=1}^{\infty}\left[1 + \left(\frac{x}{2n\pi s}\right)^2\right]\end{aligned} \tag{66}$$

has only pure imaginary zeros. ■

Corollary 10. For the n-vector model (2)

$$T_c \leqslant J/k_B n \tag{67}$$

Proof. It follows from (55) that the critical temperature for the n-vector model is bounded above by the critical temperature for the one-component model with single-spin measure given by (41). For this one-component model $\nu \in \mathfrak{M}$ and [see (56)]

$$\Phi(x) = \Gamma[(1/2)n]I_{(1/2)n-1}(x)/[(1/2)x]^{(1/2)n-1}$$
$$= \prod_{l=1}^{\infty}(1 + x^2/\alpha_{n,l}^2) \qquad (68)$$

where $\alpha_{n,l}$, $l = 1, 2, 3, \ldots$ are the positive zeros of the (unmodified) Bessel function $J_\mu(x)$ of order $\mu = (1/2)n - 1$. The required result thus follows by applying Theorem 8 to this one-component model.

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